Staying Regular?

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ALI G: *So what is the chances that me will eventually die?*
C. EVERETT KOOP: *That you will die? – 100%. I can guarantee that 100%: you will die.*
ALI G: *You is being a bit of a pessimist…*

–Ali G, interviewing the
Surgeon General, C. Everett Koop

Autobiographical back story

- Over my philosophical career I’ve been interested in various topics, but certain topics have especially gripped me…

Introduction

- I'll discuss the fluctuating fortunes of *regularity:*

If X is possible, then the probability of X is positive.

$$\Diamond X \rightarrow P(X) > 0.$$

Introduction

- I'll give many reasons to care about regularity.
- So it's important to formulate it carefully.
- I'll look at various formulations of it for subjective probability, some implausible, some more plausible.
- I'll offer what I take to be its most plausible version: a constraint that bridges *doxastic* modality and *doxastic* (subjective) probability.
- But even that will fail.

Introduction

- There will be two different ways to violate regularity
  - zero probabilities
  - no probabilities at all (probability gaps).
- Both ways create trouble for pillars of Bayesian orthodoxy:
  - the ratio formula for conditional probability
  - conditionalization, characterized with that formula
  - the multiplication formula for independence
  - expected utility theory

Introduction

- The failure of this seemingly innocuous constraint has ramifications that strike at the heart of probability theory and formal epistemology.

Regularity

If X is possible, then the probability of X is positive.

- We already had the probability axiom:

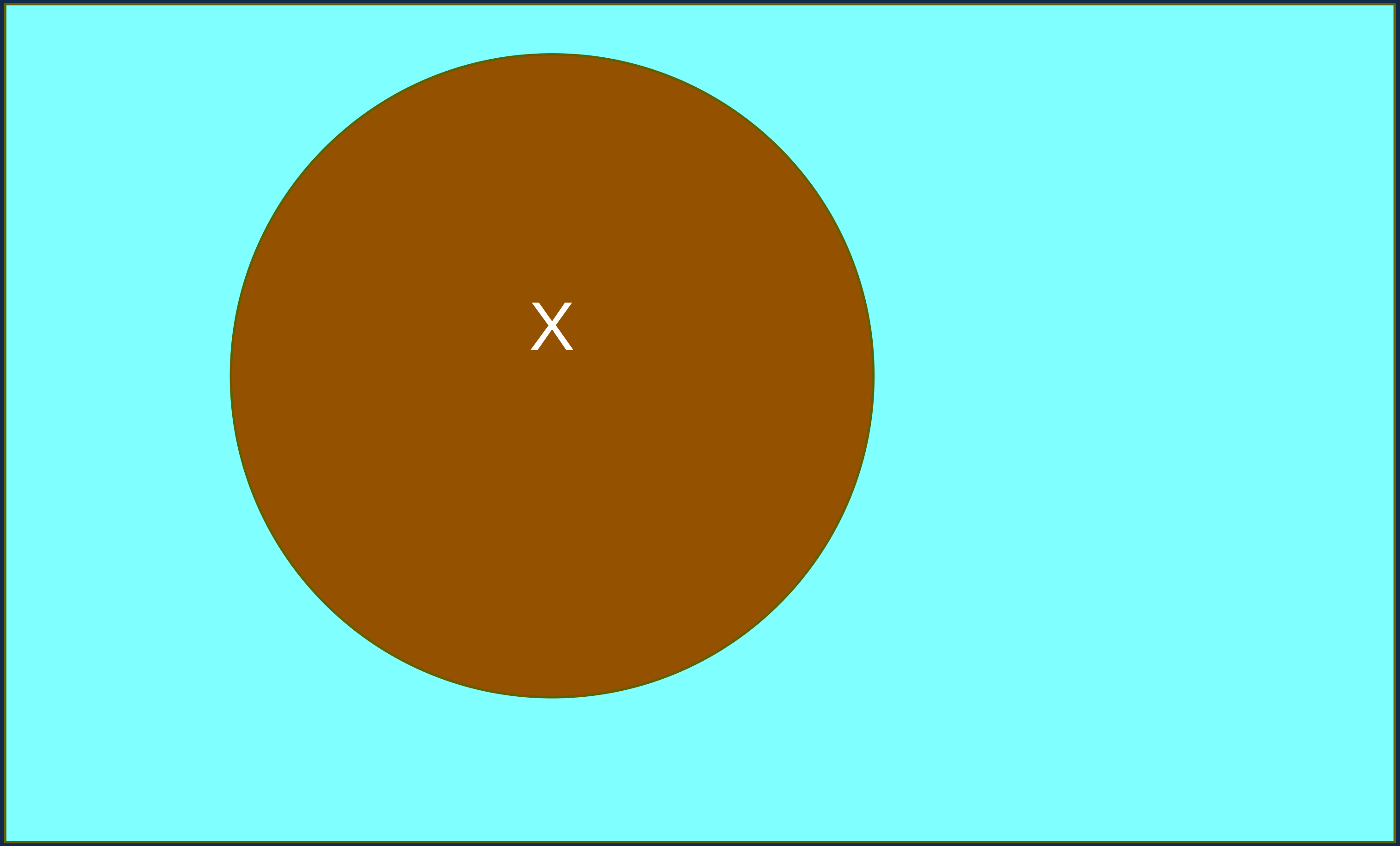
  $$P(X) \geq 0$$

- Now this constraint gets the tiniest strengthening if X is possible; the inequality becomes strict:

  $$P(X) > 0$$

  if X is possible.

- Muddy Venn diagram: no bald spots.



Regularity

- An unmnemonic name, but a commonsensical idea.
- “If it can happen, then it has a chance of happening”…

Advocates of regularity

- Regularity has been suggested or advocated by Jeffreys, Jeffrey, Carnap, Shimony, Kemeny, Edwards, Lindman, Savage, Stalnaker, Lewis, Skyrms, Appiah, Jackson, Hofweber, …

Ten reasons to care about regularity

- Regularity promises a bridge between modality and probability—a bridge that illuminates both.

Ten reasons to care about regularity

- Regularity promises a bridge between probability and truth:

  If X has probability 0, then X is impossible, hence (actually) false.

  If X has probability 1, then X is necessary, hence (actually) true.
- (No assumption of Humean supervenience.)
- If regularity fails, even this is a bridge too far!

Ten reasons to care about regularity

- Regularity may provide a bridge between traditional epistemology and Bayesian epistemology.

Ten reasons to care about regularity

- Regularity promises to illuminate rationality.
- It would provide a much-needed additional constraint on rational credence that goes beyond coherence.

Ten reasons to care about regularity

- Various Bayesian convergence results require regularity.

Ten reasons to care about regularity

- Regularity would allow us to simplify various ‘probability 1’ convergence theorems – for example, the strong law of large numbers.
- The ‘probability 1’ qualification could be removed for any regular probability function, as it would be redundant.

Ten reasons to care about regularity

- Centrepieces of synchronic Bayesian epistemology face problems when regularity fails.

Ten reasons to care about regularity

- The centrepiece of *diachronic* Bayesian epistemology – conditionalisation – faces problems without a version of regularity; yet it also conflicts with regularity.

Ten reasons to care about regularity

- Bayesian decision theory faces problems if regularity fails.
- So failures of regularity pose some of the most important problems for probability theory as a representation of uncertainty.

Ten reasons to care about regularity

- These failures motivate other representations of uncertainty – Popper functions, ranking functions, NAP, comparative probabilities…

Formulating regularity

If X is possible, then the probability of X is positive.

- This is just a schema.
- There are many senses of ‘possible’ in the antecedent…
- There are also many senses of ‘probability’ in the consequent…

Formulating regularity

- Pair them up, and we get many, many regularity conditions.
- Some are interesting, and some are not; some are plausible, and some are not.
- Focus on pairings that are definitely interesting, and somewhat plausible, at least initially.

Formulating regularity

- In the consequent, let's restrict our attention to *rational subjective* probabilities.
- If X is possible, $C(X) > 0$.
- In the antecedent? …

Formulating regularity

- Untenable:

Logical Regularity

If X is LOGICALLY possible, then C(X) > 0.

(Shimony, Skyrms)

Formulating regularity

- Problems: There are all sorts of propositions that are logically possible, but that are a priori knowable to be false, and may rationally be assigned credence 0:
  - ‘Obama is a 3-place relation’
  - ‘Clinton is the number 17’

Formulating regularity

- The probability axioms are not themselves *logically necessary*, so logical regularity curiously would require an agent to give positive credence to their falsehood.

Formulating regularity

- More plausible:

<u>Metaphysical Regularity</u>

If X is METAPHYSICALLY possible, then C(X) > 0.

Formulating regularity

- This brings us to Lewis's (1980) formulation of "regularity": "$C(X)$ is zero … only if X is the empty proposition, true at no worlds". (According to Lewis, X is metaphysically possible iff it is true at some world.)
- Lewis regards regularity in this sense as a constraint on "initial" (prior) credence functions of agents as they begin their Bayesian odysseys—Bayesian Superbabies.

Formulating regularity

- A problem for metaphysical regularity as a constraint on Superbabies: it is metaphysically possible for no thinking thing to exist, so by regularity, one must assign positive probability to this.
- But far from being rationally required, this seems to be *irrational*.
- Dutch Book argument.
- It's at least rationally *permissible* to assign probability 0 to no thinking thing existing.

Formulating regularity

- However, *doxastic* possibility seems to be a promising candidate for pairing with subjective probability.
- <u>Doxastic regularity:</u>

  If X is doxastically possible then C(X) > 0.

Formulating regularity

- We might think of a doxastic possibility for an agent as:
  - something that is compatible with what she *believes*;
  - or something that she is not certain is false;
  - or perhaps some other understanding …
  - I will speak of a doxastically *live* possibility—for short, a *live* possibility.

Formulating regularity

- So from now on I will understand regularity as:

  if X is a live possibility then C(X) > 0

- All the better that this can be understood in multiple ways. For I will argue that on any reasonable undertanding of ‘live possibility’, it is false.

Formulating regularity

- If doxastic regularity is violated, then offhand two different attitudes are conflated…
- Not just at 0, but throughout the entire [0, 1] interval.

Formulating regularity

- Doxastic regularity avoids the problems with the previous versions…

Formulating regularity

- And yet doxastic regularity appears to be untenable.

Formulating regularity

- If this version of regularity fails, then various other interesting versions will fail too. E.g.:
- Epistemic regularity:

  If X is epistemically possible, then C(X) > 0.
- This is stronger than doxastic regularity; if it fails, so does this.

Formulating regularity

- <u>Evidential regularity:</u>

  If X is not ruled out by one's evidence, then $C(X) > 0$

Two ways to be irregular

- There are two ways in which an agent’s probability function could fail to be regular:

    1) It assigns zero to some live possibility.

    2) It fails to assign *anything* to a live possibility.

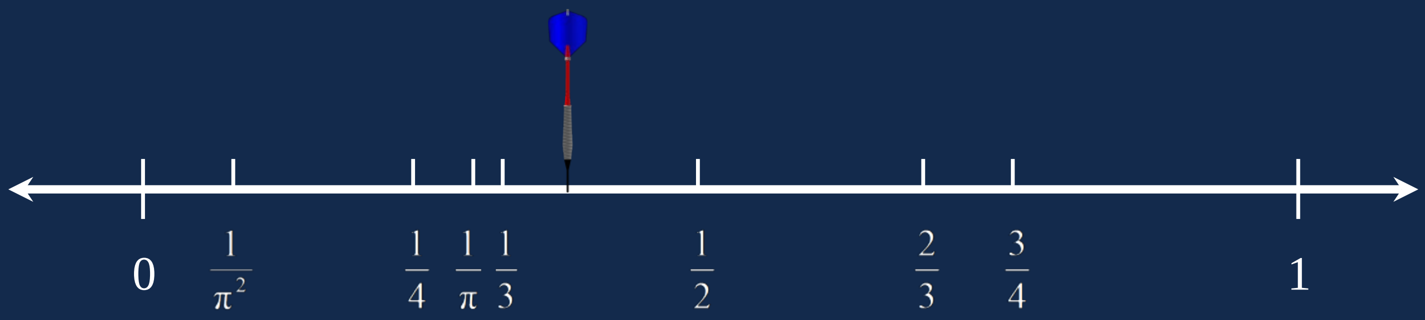
Two ways to be irregular

- Those who regard regularity as a norm of rationality must insist that all instances of 1) *and* all instances of 2) are violations of rationality.
- I will argue that there are rational instances of both 1) and 2).

Dart example

Throw a dart at random at the [0, 1] interval of the reals …

Dart example



Dart example

- Various non-empty subsets get assigned probability 0:
  - All the singletons
  - Indeed, all the finite subsets
  - Indeed, all the countable subsets
  - Even various uncountable subsets (e.g. Cantor's 'ternary set')

Dart example

- Examples like this pose a threat to regularity as a norm of rationality.
- Any landing point in [0, 1] is a live possibility for our ideal agent.

Arguments against regularity

- In order for P to be regular, there has to be a certain harmony between the *cardinalities* of P's sample space and its range.
- If the sample space is too large relative to P, regularity will be violated.

Arguments against regularity

Kolmogorov's axiomatization requires *P* to be *real*-valued. This means that any uncountable probability space is automatically irregular. (Hájek 2003).

Arguments against regularity

- It is curious that this axiomatization is restrictive on the *range* of all probability functions: the real numbers in [0,1], and not a richer set;
- yet it is almost completely permissive about their *domains:* Ω (the sample space) can be any set you like, however large, and *F* (the set of subsets that get assigned probabilities) can be any field on Ω, however large.

Arguments against regularity

- *We can apparently make the set of* <u>*contents*</u> *of an agent's thoughts as big as we like.*
- *But we limit the* <u>*attitudes*</u> *that she can bear to those contents*—the attitudes can only achieve a certain fineness of grain.
- Put a rich set of contents together with a relatively impoverished set of attitudes, and you violate regularity.

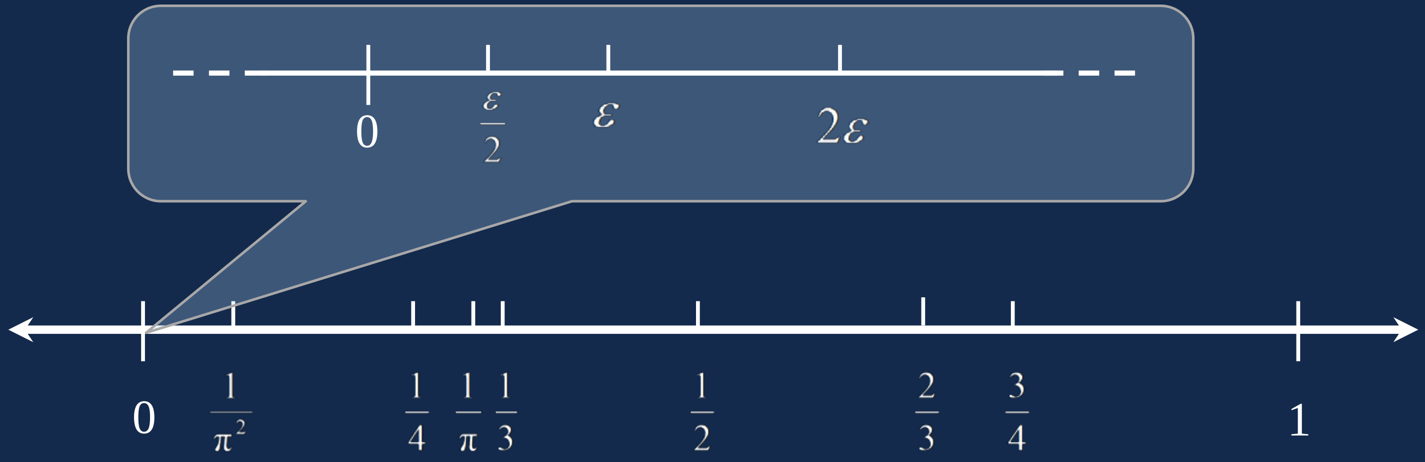
Infinitesimals to the rescue?

The friend of regularity replies: if you're going to have a rich *domain* of the probability function, you'd better have a rich *range.*

Lewis:

"You may protest that there are too many alternative possible worlds to permit regularity. But that is so only if we suppose, as I do not, that the values of the function *C* are restricted to the standard reals. Many propositions must have infinitesimal *C*-values ... (See Bernstein and Wattenberg (1969).)"

Infinitesimals to the rescue?



Infinitesimals to the rescue?

- Bernstein and Wattenberg’s article does not substantiate Lewis’ strong claim that there are too many possible worlds to permit regularity *only if C*’s values are restricted to the reals.
- Bernstein and Wattenberg show that using infinitesimals, one can give a regular probability assignment to the landing points of our fair dart throw.

Infinitesimals to the rescue?

- But that's a very specific case, with a specific cardinality!
- Lewis himself thinks that the cardinality of the set of possible worlds is greater than that (at least beth-2).
- We need a similar result that holds if the set of possibilities has higher cardinality than that of the real interval [0, 1].
- Indeed, the set of doxastic possibilities may well be *a proper class! …*

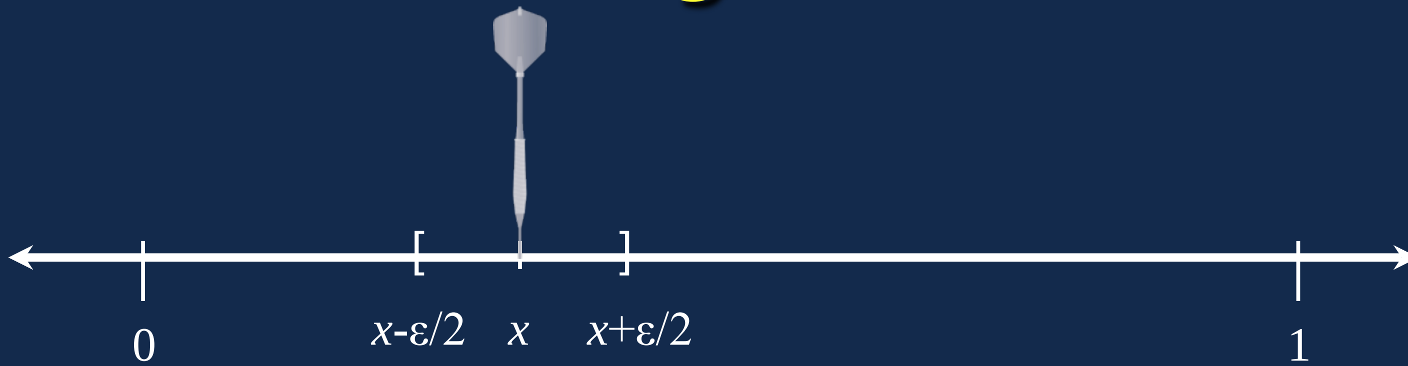
Arguments against regularity, even allowing infinitesimals

- I conjectured that a version of the cardinality problem would always arise.
- Pruss proved it: if the cardinality of Ω is greater than that of the range of P, then regularity fails.
- No symmetry assumption is needed – cardinalities do all the work.

Arguments against regularity, even allowing infinitesimals

- But if we add a symmetry assumption, we have another, more intuitive argument.
- We can scotch regularity even for a hyperreal-valued probability function by correspondingly enriching the space of possibilities…
- The dart is thrown at the [0, 1] interval of the *hyperreals*.

Arguments against regularity, even allowing infinitesimals



Not to scale!

- x is strictly contained within nested intervals of width ε, for each infinitesimal ε. The probability of each interval is its width, ε. (This assumption can be somewhat weakened.)
- So the point's probability is bounded above by *all* these ε, and thus it must be smaller than all of them—i.e. 0.

Arguments against regularity, even allowing infinitesimals

I envisage a kind of arms race:

- We scotched regularity for real-valued probability functions with sufficiently large domains (uncountable).
- The friends of regularity fought back, enriching their ranges: making them hyperreal-valued.
- The enemy of regularity counters by enriching the domain.
- And so it goes.
- By Pruss's result, the enemy can always win (for anything that looks like Kolmogorov's probability theory).

Arguments against regularity, even allowing infinitesimals

- Could we tailor the range of the probability function to the domain, for each particular application? (Like the general of a defense force …)
- The trouble is that in a Kolmogorov-style axiomatization the commitment to the range of P comes *first*…
- On the tailoring approach, a probability function is a mapping from F to …—well, to *what*?
- What will the additivity axiom look like?
- In any case, this 'wait and see' approach is quite a departure from Kolmogorov.

Arguments against regularity, even allowing infinitesimals

- On a Kolmogorov-style approach, there will always be an Ω that will have non-empty subsets assigned probability 0.

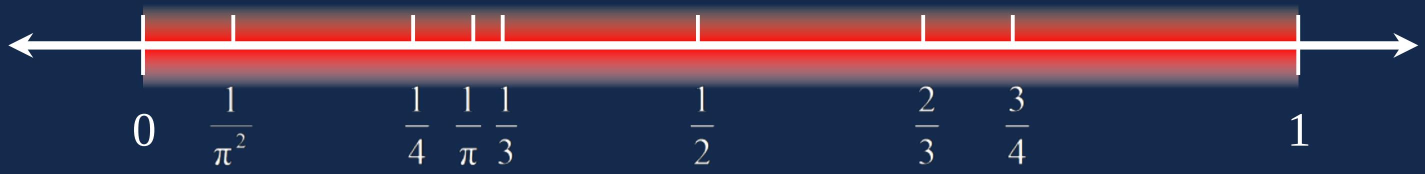
Doxastically possible credence gaps

- I will argue that you can rationally have credence gaps.

Examples of doxastically possible credence gaps

- Non-measurable sets

Dart example



- Certain subsets of Ω—so-called *non-measurable* sets—get no probability assignments whatsoever.

Examples of doxastically possible credence gaps

- Chance gaps
- The Principal Principle says (roughly!!):

  your credence in X, *conditional* on it having chance x, should be x:

  $$C(X \mid \text{chance}(X) = x) = x.$$

Examples of doxastically possible credence gaps

- A relative of the Principal Principle? Roughly:

  your credence in *X*, *conditional* on it being a chance gap, should be gappy:

  $C(X \mid \text{chance}(X) \text{ is } \textit{undefined}) \text{ is } \textit{undefined}.$

- All I need is that rationality *sometimes permits* your credence to be gappy for a hypothesized chance gap.

Examples of doxastically possible credence gaps

- There are arguably various examples of chance gaps:
  - Chance statements themselves
  - Cases of indeterminism without chances: Earman’s space invaders, Norton’s dome (Eagle)

Examples of doxastically possible credence gaps

- One’s own free choices
- Kyburg, Gilboa, Spohn, Levi, Briggs, Liu and Price contend that when I am making a choice, I must regard it as free. In doing so, I *cannot* assign probabilities to my acting in one way rather than another (even though onlookers may be able to do so).
- “Deliberation crowds out prediction”—or better, it crowds out probability.

Examples of doxastically possible credence gaps

- To be sure, these cases of probability gaps are controversial.
- But these authors are committed to there being credence gaps, and thus violations of regularity.
- All I need is that it is *permissible* for them to be credence gaps.

Ramifications of irregularity for Bayesian epistemology and decision theory

- I have argued for two kinds of counterexamples to regularity: rational assignments of zero credences, and rational credence gaps, for doxastic possibilities.
- I now want to explore some of the unwelcome consequences these failures of regularity have for traditional Bayesian epistemology and decision theory.

Problems for the conditional probability ratio formula

- The ratio analysis of conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

... provided P(B) > 0

Problems for the conditional probability ratio formula

- What is the probability that the dart lands on ½, *given* that it lands on ½?
- 1.
- But the ratio formula cannot deliver that result, because P(dart lands on ½) = 0.

Problems for the conditional probability ratio formula

- Gaps create similar problems.
- Take your favorite probability gap, G.
- The probability of G, given G, is 1.
- But the ratio formula cannot deliver that result, because

  P(G) is undefined.

Problems for the conditional probability ratio formula

- We need a more sophisticated account of conditional probability.
- I advocate taking conditional probability as primitive (in the style of Popper and Rényi).

Problems for conditionalization

- The zero-probability problem for the conditional probability formula quickly becomes a problem for the updating rule of *conditionalization*, which is defined in terms of it.
- Suppose the agent learns evidence *E.*

$$P_{new}(X) = P_{old}(X \mid E) \quad \text{(provided } P_{old}(E) > 0\text{)}$$

Problems for conditionalization

- Suppose you *learn* that the dart lands on ½. What should be your *new* probability that the dart lands on ½?
- 1.
- But

  $$P_{old}(\text{dart lands on } ½ \mid \text{dart lands on } ½)$$

  is undefined, so conditionalization (so defined) cannot give you this advice.

Problems for conditionalization

- Gaps create similar problems.
- Suppose you *learn* that G. What should be your *new* probability for G?
- 1.
- But

  $$P_{\text{old}}(G \mid G)$$

  is undefined, so conditionalization cannot give you this advice.

Problems for conditionalization

- We need a more sophisticated account of conditionalization.
- Primitive conditional probabilities to the rescue!

Problems for independence

- We want to capture the idea of A being probabilistically uninformative about B.
- A and B are said to be *independent* just in case

$$P(A \cap B) = P(A)\, P(B).$$

Problems for independence

- According to this account of probabilistic independence, anything with probability 0 is independent of *itself*:

  If $P(X) = 0$, then $P(X \cap X) = 0 = P(X)P(X)$.

- But identity is the ultimate case of (probabilistic) *d*ependence.

Problems for independence

- Suppose you are wondering whether the dart landed on ½. *Nothing* could be more informative than your learning: the dart landed on ½.
- But according to this account of independence, the dart landing on ½ is independent of the dart landing on ½!

Problems for independence

- Gaps create similar problems.
- Suppose you are wondering whether G. *Nothing* could be more informative than your learning: G.
- But there is no verdict from this account of independence.

Problems for independence

- We need a more sophisticated account of independence – e.g. using primitive conditional probabilities.
- Branden Fitelson and I have been working on this!

Problems for expected utility theory

- Arguably the two most important foundations of decision theory are the notion of *expected utility*, and *dominance reasoning*.

Problems for expected utility theory

- And yet probability 0 propositions apparently show that expected utility theory and dominance reasoning can give conflicting verdicts.

Problems for expected utility theory

- Suppose that two options yield the same utility except on a proposition of probability 0; but if that proposition is true, option 1 is far superior to option 2.

Problems for expected utility theory

- You can choose between these two options:
  - Option 1: If the dart lands on 1/2, you get a million dollars; otherwise you get nothing.
  - Option 2: You get nothing.

Problems for expected utility theory

- Expected utility theory apparently says that these options are equally good: they both have an expected utility of 0.
- But dominance reasoning says that option 1 is strictly better than option 2. Which is it to be?
- I say that option 1 is better.
- I think that this is a counterexample to expected utility theory as it is usually interpreted.
- Both evidential and causal.
- (To be sure, there are replies …)

Problems for expected utility theory

- Gaps create similar problems.
- You can choose between these two options:
  - Option 1: If G, you get a million dollars; otherwise you get nothing.
  - Option 2: You get nothing.

Problems for expected utility theory

- Expected utility theory goes silent.
- I say that option 1 is better.
- We need a more sophisticated decision theory.

Conclusion

- Irregularity makes things go bad for the orthodox Bayesian; that is a reason to insist on regularity.
- The trouble is that regularity appears to be untenable.
- I focused on doxastic regularity, but other interesting regularities will meet similar downfalls.
- I think, then, that irregularity is a reason for the orthodox Bayesian to become unorthodox.

Conclusion

- I have advocated replacing the orthodox theory of conditional probability, conditionalization, and independence with alternatives based on Popper/Rényi functions. Expected utility theory appears to be similarly in need of revision.

Conclusion

- And then there are some possibilities that *really should* be assigned zero probability …

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Ten reasons to care about regularity

- Regularity may provide a bridge between logic and probability.
- Failure of regularity is a thorn in the side of probabilistic semantics for logic. Probabilistic notions of entailment, incompatibility are poor surrogates for their logical counterparts.