Logics of Belief based on Logics of Information

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What kind of agents we have in mind, and what aspects of belief we want to model?

- A prototypical agent — a scientist (cf. scientific or rational scepticism),
- working with collections of data — those might be incomplete and inconsistent.
- The agent (e.g. by weighting the available evidence) eventually accepts some *available data* as beliefs,
- but only confirmed data might be accepted (certified belief).

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- A background propositional logic to model collections of data—information states — (containing a reasonable negation),
- collections of data (information states or evidence states) are modeled as theories.
- Agents allow for some information states to act as reliable sources of confirmation for a given state.
- Modal part consists of an epistemic diamond operator of confirmed belief.

Examples: substructural epistemic logics (over dFLe)

Language

$$\alpha ::= \ p \mid t \mid \alpha \otimes \alpha \mid \alpha \to \alpha \ \mid \top \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \langle k \rangle \alpha \mid \langle b \rangle \alpha$$

interpreted over frames $F = (X, \leq, R, L, C, S^k, S^b)$ as (formulas are interpreted by upsets):

- $x \Vdash \neg\alpha$ iff $\forall y \ (xCy \ \longrightarrow \ y \nVdash \alpha)$
- $x \Vdash \langle k \rangle \alpha$ iff $\exists s \ (sS^k x \wedge s \Vdash \alpha)$
- $x \Vdash \langle b \rangle \alpha$ iff $\exists s \ (sS^b x \wedge s \Vdash \alpha)$

We read $sS^k x$ as s is a reliable source confirming knowledge in x. Similarly for belief.

Properties of the source relations

- Sources for belief are mutually compatible (do not contradict each other). Sources of knowledge are compatible with the current state.
- Sources are self-compatible (therefore consistent).
- $S^k \subseteq \leq$ implies that if α is known, it is satisfied in the current state.
- $S^k \subseteq S^b$: knowledge implies belief

Beliefs are mutually consistent. Knowledge is consistent with the current information state, and knowledge (due to persistency of formulas) is factive.

Axioms and corresponding classes of frames

$\langle k \rangle$ and $\langle b \rangle$ are monotone normal diamond modalities. Moreover we may consider (some of) the following axioms:

Axiom or rule	condition
$\langle k \rangle\alpha \to \alpha$	$sS^k x \longrightarrow s \leq x$
$\langle k \rangle\alpha \to \langle b \rangle\alpha$	$sS^k x \longrightarrow sS^b x$
$\langle b \rangle\alpha \wedge \langle b \rangle\neg\alpha \to \bot$	$sS^b x \wedge s' S^b x \longrightarrow sCs'$
$\langle b \rangle(\alpha \wedge \neg\alpha) \to \bot$	$sS^b x \longrightarrow sCs$
$\langle k \rangle\alpha \wedge \neg\alpha \to \bot$	$sS^k x \longrightarrow sCx$
$\langle k \rangle\alpha \to \langle k \rangle\langle k \rangle\alpha$	$sS^k x \longrightarrow \exists s' \ (sS^k s' S^k x)$
$\langle b \rangle\alpha \to \langle b \rangle\langle b \rangle\alpha$	$sS^b x \longrightarrow \exists s' \ (sS^b s' S^b x)$
$\langle b \rangle\alpha \to \langle b \rangle\langle k \rangle\alpha$	$sS^b x \longrightarrow \exists s' \ (sS^k s' S^b x)$
$\langle k \rangle\alpha \wedge \langle k \rangle\beta \to \langle k \rangle(\alpha \wedge \beta)$	$sS^k x \wedge tS^k x \longrightarrow \exists v \ (vS^k x \wedge s, t \leq v)$
$\vdash \alpha / \vdash \langle k \rangle\alpha$	$(\forall x \in L)(\exists s \in L) \ sS^k x$

Examples: Relevant epistemic logic

Frames for relevant logic in style of Restall's book on substructural logic, the source relation satisfying:

- $sSx \longrightarrow s \leq x$
- $sSx \longrightarrow sCx$

M. Bílková, O. Majer, M. Peliš and G. Restall. Relevant agents. AiML 2010.
T. Childers, O. Majer and P. Milne. The relevant logic of scientific discovery. In progress.

Examples: Intuitionistic epistemic logic

(i) From the standard semantics of intuitionistic logic: for a poset $(X, \leq)$, put $L = X$, let S^k to be any monotone relation satisfying $S^k \subseteq \leq$, and define the remaining relations as follows:

$$\begin{array}{rcl} Rxyz & \text{iff} & x \leq z \text{ and } y \leq z \\ Cxy & \text{iff} & \exists z(x \leq z \text{ and } y \leq z) \end{array}$$

The modality is not trivial ($\alpha \nvdash \langle k \rangle \alpha$), and neither it commutes with the conjunction nor distributes to the implication.

Examples: Intuitionistic epistemic logic

(ii) Consider $(X, \leq)$ to be a rooted tree with the root r. Put $rS^k x$ for all $x \in X$ (the root r is a universal source). In this class of frames, $\langle k \rangle$ commutes with conjunction, distributes to implication, positive introspection axiom becomes valid, as well as negative introspection axiom.

Results

- A concept of confirmed belief or knowledge can be modeled as a diamond modality over suitable semantics, e.g. relational semantics for substructural logics.
- Strong completeness, FMP via filtration.
- Structural (display) proof theory, cut elimination.
- Common knowledge and common belief, with infinitary, strongly complete, proof systems.

Problems and solutions

- Q: Why a diamond modality?
- A: We model confirmed belief and knowledge. Moreover, we can naturally arrive at such a modality from a monotone neighbourhood box modality.
- Q: Why a normal diamond? Knowledge distributing over the disjunction is counter-intuitive.
- A: Consider another semantics of disjunction, under which the information states are not necessarily closed; or, switch to neighborhood semantics.
- Q: What do we mean if we say that beliefs are consistent?
- A: Possibly different things (to avoid explosion, or to avoid various contradictions).

Semi-lattice frames

- Frames based on (distributive) meet semi-lattices instead of posets,
- canonical frames based on theories rather then prime theories,
- Disjunction is interpreted modally using the meet. This allows to control its distributivity properties.

cf.
V. Punčochář. Algebras of Information States. Journal of Logic and Computation, Volume 27, Issue 5, 2017.
V. Punčochář. Knowledge is a diamond. WOLLIC 2017.
Remark: Implicit also in semantics for non-distributive substructural logics based on polarity frames.

Semi-lattice frames

A *frame* $F = (X, \leq, \wedge, \tau, C, S)$, where

- $(X, \leq, \wedge, \top)$ is a meet semi-lattice of information states, where formulas are to be interpreted as filters,
- the frame may but need not satisfy

$$x \wedge y \leq z \longrightarrow \exists x', y'(x \leq x' \;\&\; y \leq y' \;\&\; x' \wedge y' = z)$$

[*distributivity*]

- $\top$ a top element: consequently, $\alpha \vdash \alpha \vee \beta$.

Semi-lattice frames

A *frame* $F = (X, \leq, \wedge, \tau, C, S)$, where

- C is a symmetric binary *compatibility* relation on X, with $\neg tCx$ and:

$$x' \leq x \; C \; y \geq y' \quad \longrightarrow \quad x' \; C \; y'$$

[*monotonicity*]

$$x \wedge y \; C \; z \quad \longrightarrow \quad x \; C \; z \text{ or } y \; C \; z$$

[*regularity*]

(consequently, negation creates persistent formulas, and $\neg\alpha \wedge \neg\beta \vdash \neg(\alpha \vee \beta)$).

Semi-lattice frames

A *frame* $F = (X, \leq, \wedge, \tau, C, S)$, where

- S is a binary *source* relation on X:

$$x\,S\,y \leq y' \quad \longrightarrow \quad x\,S\,y'$$

[*monotonicity*]

$$x\,S\,z \;\&\; x'\,S\,u \quad \longrightarrow \quad x \wedge x'\,S\,z \wedge u$$

[*regularity*]

(consequently, $\langle b \rangle$ creates persistent and regular formulas).

Interpreting the language

We call a proposition $a \subseteq X$ persistent iff a is closed upwards and regular iff a is $\wedge$ closed. Persistent and regular propositions correspond to filters on X.
Language

$$\alpha ::= \; p \mid \top \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \langle b \rangle \alpha$$

A valuation is a map $V : \mathrm{Prop} \longrightarrow \mathcal{F}X$

- $x \Vdash p$ iff $x \in V(p)$
- $x \Vdash \top$ and $x \Vdash \bot \longleftrightarrow x = \tau$
- $x \Vdash \alpha \wedge \beta$ iff $x \Vdash \alpha$ and $x \Vdash \beta$
- $x \Vdash \alpha \vee \beta$ iff $\exists y, z \; (y \wedge z \leq x \;\&\; y \Vdash \alpha \;\&\; z \Vdash \beta)$
- $x \Vdash \neg\alpha$ iff $\forall y \; (xCy \longrightarrow y \nVdash \alpha)$
- $x \Vdash \langle b \rangle \alpha$ iff $\exists s \; (sSx \;\&\; s \Vdash \alpha)$

If all the relations satisfy the monotonicity conditions, all formulas are persistent. If all the relations moreover satisfy the regularity conditions, all formulas denote filters.

Axioms, and corresponding classes of frames

- $\alpha \vdash \beta$ is valid in a frame X, iff $\forall x \in X(x \Vdash \alpha$ implies $x \Vdash \beta)$.
- $\Gamma \vdash \alpha$ iff for some finite $\Gamma' \subseteq \Gamma$, $\bigwedge \Gamma' \vdash \alpha$ is provable in the following system L:

$\bot \vdash \alpha$	$\alpha \vdash \top$	$\alpha \vdash \alpha$
$\alpha \vdash \alpha \vee \beta$	$\alpha \vdash \alpha \vee \beta$	$\alpha \vdash \chi, \beta \vdash \chi \ / \ \alpha \vee \beta \vdash \chi$
$\alpha \wedge \beta \vdash \alpha$	$\alpha \wedge \beta \vdash \beta$	$\chi \vdash \alpha, \chi \vdash \beta \ / \ \chi \vdash \alpha \wedge \beta$
$\alpha \vdash \neg\beta \ / \ \beta \vdash \neg\alpha$		$\alpha \vdash \beta, \beta \vdash \chi \ / \ \alpha \vdash \chi$

Axioms, and corresponding classes of frames

plus additional axioms:

Axiom	condition
$\langle b\rangle\alpha \vdash \alpha$	$sSx \to s \leq x$
$\langle b\rangle\alpha \wedge \neg\alpha \vdash \bot$	$sSx \to sCx$
$\langle b\rangle\alpha \wedge \langle b\rangle\neg\alpha \vdash \bot$	$sSx \ \&\ s'Sx \to sCs'$
$\langle b\rangle\alpha \vdash \langle b\rangle\langle b\rangle\alpha$	$sSx \to \exists s'\ (sSs'Sx)$
$\langle b\rangle(\alpha \vee \beta) \vdash \langle b\rangle\alpha \vee \langle b\rangle\beta$	$x \wedge ySz \to \exists u, v(xSu, ySv \ \&\ u \wedge v \leq z)$
$\langle b\rangle\alpha \wedge \langle b\rangle\beta \vdash \langle b\rangle(\alpha \wedge \beta)$	$sSx \wedge tSx \longrightarrow \exists v\ (vSx \ \&\ s, t \leq v)$
$\neg\alpha \wedge \neg\beta \vdash \neg(\alpha \vee \beta)$	$x \wedge yCz \to xCz$ or yCz
$\neg\neg\alpha \vdash \alpha$	$x = \tau \vee (\exists \max.y)xCy$
$\alpha \wedge (\beta \vee \chi) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \chi)$	$x \wedge y \leq z \to \exists u \geq x, v \geq y(u \wedge v = z)$
$\neg\top \vdash \bot$	$x = \tau\ \vee (\exists y)xCy$
$\alpha \wedge \neg\alpha \vdash \bot$	$x = \tau\ \vee xCx$

We shall come back to this to address the consistency of beliefs.

Completeness via canonical model

Theorem (Strong Completeness)

The axiomatization (L + Ax) is strongly complete with respect to the class of corresponding epistemic frames.

$$\Gamma \nvdash \alpha \text{ implies } \Gamma \nvDash_{\mathcal{F}(Ax)} \alpha$$

Proof — the canonical model construction. Canonical states = theories with the intersection, ordered by inclusion, canonical relations defined as usual. All axioms listed above are canonical.

Variability of the semi-lattice semantics

- We may relax persistence/regularity
- It is possible to add

$$x \Vdash \alpha \sqcup \beta \text{ iff } x \Vdash \alpha \text{ or } x \Vdash \beta$$

  and obtain an inquisitive logic (not every formula is regular, requires a multitype proof theory).
- It is possible to extend the propositional base to a substructural one, e.g. FLe., and to vary the properties of the negation.

Consistency of beliefs

For sets $\Gamma_x = \{\alpha \mid x \Vdash \langle b \rangle \alpha\}$ we may want the following:

- $\Gamma_x \nvdash \bot$
- $\Gamma_x \nvdash \neg\alpha$ for $\alpha \in \Gamma_x$ (and $\Gamma_x \nvdash \alpha$ for $\neg\alpha \in \Gamma_x$)
- $\Gamma_x \nvdash \alpha \wedge \neg\alpha$ for all α

Example: The factive and strongly consistent notion of knowledge we considered previously avoids the first two, but not the third one (unless negation is fully de Morgan).
But consistency axioms of belief which need not be factive are too weak to ensure any of those.

Consistency of beliefs

For sets $\Gamma_x = \{\alpha \mid x \Vdash \langle b \rangle \alpha\}$ we may moreover want the following:

- $\Gamma_x \nvdash \bigvee \overline{\alpha_i}$ for any $\alpha_i \in \Gamma_x$
- $\Gamma_x \nvdash \bigvee (\alpha_i \wedge \neg \alpha_i)$ for any α_i

Example: The factive and strongly consistent notion of knowledge now need not avoid those two.

Consistency of beliefs

- $\Gamma_x \nvdash \bot$
- $\Gamma_x \nvdash \neg\alpha$ for $\alpha \in \Gamma_x$ (and $\Gamma_x \nvdash \alpha$ for $\neg\alpha \in \Gamma_x$)
- $\Gamma_x \nvdash \alpha \wedge \neg\alpha$ for all α

are respectively characterized by conditions:

- $s_1 \dots s_n S^b x \longrightarrow \exists t(s_1 \dots s_n \leq t \;\&\; t \neq \tau)$
- $s_1 \dots s_n S^b x \longrightarrow \exists t(s_1 \dots s_n \leq t \;\&\; s_1 \dots s_n C t)$
- $s_1 \dots s_n S^b x \longrightarrow \exists t(s_1 \dots s_n \leq t \;\&\; tCt)$

and completely axiomatized by rules, e.g.

$$\frac{\alpha_1, \dots, \alpha_n \vdash \bot}{\langle b \rangle \alpha_1, \dots, \langle b \rangle \alpha_n \vdash \bot}$$

Consistency of beliefs

- $\Gamma_x \nvdash \bigvee \overline{\alpha_i}$ for any $\alpha_i \in \Gamma_x$
- $\Gamma_x \nvdash \bigvee(\alpha_i \wedge \neg\alpha_i)$ for any α_i

are respectively characterized by conditions:

- $s_1 \ldots s_n S^b x \longrightarrow \exists t \in MIR(X)(s_1 \ldots s_n \leq t \ \& \ s_1 \ldots s_n Ct)$
- $s_1 \ldots s_n S^b x \longrightarrow \exists t \in MIR(X)(s_1 \ldots s_n \leq t \ \& \ tCt)$

Neighborhood frames

We replace the source relation by a neighborhood function:

$$S : X \longrightarrow \mathbb{L}\mathbb{U}X,$$

and interpret belief as:

$$x \Vdash \langle b \rangle \alpha \text{ iff } \exists Y \in S(x)\ \forall y \in Y\ \ y \Vdash \alpha.$$

plus possibly additional conditions, like:

- $x \in S(x)$ characterizes factivity $\langle b \rangle \alpha \vdash \alpha$
- consistency conditions look like e.g.:

$$Y_1 \dots Y_n \in S(x) \longrightarrow \exists t (t \in \bigcap Y_i \ \&\ \forall i\ \exists y_i (y_i \in Y_i \ \&\ y_i C t)).$$

$$Y_1 \dots Y_n \in S(x) \longrightarrow \exists t (t \in \bigcap Y_i \ \&\ tCt).$$

Box or Diamond?

- Starting with a notion of possible worlds, we can define information states as subsets of possible worlds.
- Extending ideas of Vít Punčochář, we can relate logics of possible worlds and corresponding logics of information states (truth vs. assertibility).
- We can start with monotone modal logic and its neighborhood semantics, and the result will fall under our current framework:

Box or Diamond? Example of a semilattice frame

- Consider a monotone neighborhood model $(W, B : PW \longrightarrow PW)$, where $||\Box\alpha|| = B||\alpha||$
- define a frame $(PW, \supseteq, \emptyset)$ with a relation:

$$xSy \ \equiv_{df} \ B(x) \supseteq y$$

$$xCy \ \equiv_{df} x \not\subseteq \overline{y}$$

- put $\qquad x \Vdash p \ \Leftrightarrow \ x \subseteq ||p||$
- Then $\alpha \in (\wedge, \vee, \neg, \Box)$ translates to $\alpha^* \in (\wedge, \vee, \neg, \langle b \rangle)$:

$$||\alpha|| \subseteq ||\beta|| \ \text{ iff } \ \alpha^* \vdash \beta^*.$$

Groups and Common belief

Common belief for a group $G \subseteq I = \{1 \dots n\}$ can be defined via iterating "everybody believes that...". To list a few possibilities:

- $\bigwedge_{i \in G} \langle i \rangle \alpha$
- $\langle G \rangle \alpha$ interpreted via a relation S_G with $G \subseteq H \longrightarrow S_H \subseteq S_G$
  $\langle H \rangle \alpha \vdash \langle G \rangle \alpha$
- $\langle G \rangle \alpha$ interpreted via S_G with $S_{G \cup H} = S_G \cap S_H$
- $\langle G \rangle \alpha$ interpreted via S_G with $S_{G \cup H} = S_G \wedge S_H$
  $\langle H \rangle \alpha \wedge \langle G \rangle \beta \vdash \langle H \cup G \rangle \alpha \vee \beta$

Infinitary proof theory for iterative Common belief

- denote $\bigwedge_{i \in G} \langle i \rangle \alpha$, or $\langle G \rangle \alpha$ by $\Diamond \alpha$. Finite approximations of $C\alpha$:

$$C^1\alpha = \Diamond\alpha, \quad C^2\alpha = \Diamond(\alpha \wedge \Diamond\alpha), \quad C^3\alpha = \Diamond(\alpha \wedge \Diamond(\alpha \wedge \Diamond\alpha)), \ldots$$

- adopt axioms

$$C\alpha \vdash C^n\alpha \qquad C^{n+1}\alpha \vdash C^n\alpha$$

  and an infinitary rule

$$\{C^n\alpha \mid n \in N\} \vdash C\alpha$$

Strong completeness for C: consequence relation

- The resulting consequence relation $\Gamma \vdash \delta$ satisfies identity and monotonicity (weakening),
- and is closed under Infinitary Cut:

$$\frac{\Gamma, \{\beta_i \mid i \in I\} \vdash \delta \qquad \{\Gamma \vdash \beta_i \mid i \in I\}}{\Gamma \vdash \delta}$$

- but for any box-type operator (meet-preserving) $\circ$ we have to ensure:

$$\frac{\Gamma \vdash \varphi}{\circ\Gamma \vdash \circ\varphi}$$

- in our case, there is none. It could be e.g. the implication in which case we close the infinitary rule under pre-fixing. (Used later in valuation lemma).

Strong completeness for C

Theorem (Strong Completeness)

The axiomatization is strongly complete with respect to the class of corresponding epistemic frames.

$$\Gamma \nvdash \delta \text{ implies } \exists(F, V), x \ (x \Vdash \Gamma \ \& \ x \nVdash \delta)$$

Proof — the canonical model construction. Canonical states = theories with the intersection, ordered by inclusion, canonical relations defined as usual.

In the distributive setting we can consider an alternative canonical model using poset of prime theories, i.e. meet irreducible elements in the current model. S would be a neighborhood relation.

Strong completeness for C: Pair extension lemma

In the distributive setting we can build an alternative canonical model using prime theories:

- $\langle \Gamma; \Delta \rangle$ is a pair iff $\Gamma \nvdash \Delta$ (Γ proves no finite disjunction of Δ)
- in the finitary case, each pair can be extended to a full pair ($\Gamma \cup \Delta = \mathcal{L}$), where Γ is a prime theory,
- in our infinitary case it can certainly be done for finite Δ.
- To ensure that (a countable) union of a chain of pairs is again a pair, one has to modify the construction! In case $\alpha_i = C\varphi$ and $\Gamma_i, C\varphi \vdash \Delta_i$,

  $$\langle \Gamma_{i+1}; \Delta_{i+1} \rangle = \langle \Gamma_i; C\varphi, C^n\varphi, \Delta_i \rangle,$$

  (e.g. the least n for which this is a pair.)

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THANK YOU!