

Semi-completeness

– a uniform algebraic approach to cut elimination

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1. Aim of my talk

Cut elimination is one of the most important syntactic properties in sequent systems.

- A standard way of showing cut elimination is proof-theoretic. It consists of combinatorial analysis of proof structures, with a constructive procedure for eliminating each application of cut rule, using double induction.
- For some time, I have been trying to understand connections between algebraic proofs and model-theoretic proofs (i.e. semantical proofs using Kripke frames) of cut elimination.

What I am going to show here is that the idea introduced by S. Maehara in [Mae] will provide a uniform framework for understanding various semantical proofs of cut elimination.

- S. Maehara (1991): *Lattice-valued representation of the cut elimination theorem*, [Mae] Tsukuba J. of Math. 15.
- For further details of my talk, see
HO, *A unified algebraic approach to cut elimination via semi-completeness*, in: Philosophical Logic: Current Trends in Asia, to appear.

We assume that each sequent is an expression of the form $\Gamma \Rightarrow \Delta$, where both Γ and Δ are *multisets* of formulas. In particular, we take up two sequent systems in the following explanation:

- the system **GS4** for modal logic **S4** which has the following rules for $\Box$;

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box \alpha, \Gamma \Rightarrow \Delta} (\Box \Rightarrow) \qquad \frac{\Box \Gamma \Rightarrow \alpha}{\Box \Gamma \Rightarrow \Box \alpha} (\Rightarrow \Box 1)$$

Here, $\Box \Gamma$ denotes the sequence of formulas $\Box \alpha_1, \dots, \Box \alpha_m$ when Γ is $\alpha_1, \dots, \alpha_m$.

- the multiple-succedent sequent system **LJ'** (known also as **G3im**) for intuitionistic logic, whose rules $(\Rightarrow \rightarrow)$ and $(\Rightarrow \neg)$ of **LJ'** are restricted to the following form, resp.;

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\Rightarrow \rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg \alpha} (\Rightarrow \neg)$$

2. How semantical proofs will go

Let $\mathbf{S}^-$ be the system obtained from a sequent system $\mathbf{S}$ by deleting cut rule. Cut elimination of $\mathbf{S}$ says that for any sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$, (1) is equivalent to (3);

- ① $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{S}$,
- ③ $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{S}^-$.

2. How semantical proofs will go

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- ① $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{S}$,
- ② ? ? ?
- ③ $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{S}^-$.

We want to find a reasonable *semantical condition* (2) between (1) and (3).

2.1. Maehara's approach

Let $\mathbf{A}$ be any modal algebra and Z be a nonempty subset of the set Ω of all modal formulas which is **closed under subformulas**. (You may always take Ω for Z in the following, for the sake of simplicity.) Here,

- an algebra $\mathbf{A} = \langle A, \cap, \cup, ', 1, \Box \rangle$ is a modal algebra, if $\langle A, \cap, \cup, 1, ' \rangle$ is a Boolean algebra and $\Box$ is a unary operator on A satisfying $\Box 1 = 1$, and $\Box(a \cap b) = \Box a \cap \Box b$ for all $a, b \in A$.

2.2. Quasi-valuations

A pair (k, K) of mappings k and K from Z to A is a **quasi-valuation** over Z on $\mathbf{A}$, if it satisfies the following conditions;

- $k(\alpha) \leq K(\alpha)$ for $\alpha \in Z$,
- $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
- $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
- $k(\neg\alpha) \leq K(\alpha)'$ and $k(\alpha)' \leq K(\neg\alpha)$ for $\neg\alpha \in Z$,
- $k(\Box\alpha) \leq \Box k(\alpha)$ and $\Box K(\alpha) \leq K(\Box\alpha)$ for $\Box\alpha \in Z$.

Quasi-valuations can be defined also on other algebras, e.g. Heyting algebras and residuated lattices in general.

For example, a pair of mappings k and K from Z to a Heyting algebra A is a **quasi-valuation** on a Heyting algebra $\mathbf{A}$ if it satisfies the following conditions.

- $k(\alpha) \leq K(\alpha)$ for $\alpha \in Z$,
- $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
- $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
- $k(0) = 0_A$,
- $k(\alpha \rightarrow \beta) \leq K(\alpha) \rightarrow k(\beta)$ and $k(\alpha) \rightarrow K(\beta) \leq K(\alpha \rightarrow \beta)$ for $\alpha \rightarrow \beta \in Z$.

When $k(\alpha) = K(\alpha)$ for every $\alpha \in Z$, the mapping K is no other than a usual **valuation** (over Z) on $\mathbf{A}$.

Lemma (quasi-valuation lemma)

Suppose that f is a valuation and (k, K) is a quasi-valuation over Z on $\mathbf{A}$, respectively, such that $k(p) \leq f(p) \leq K(p)$ for every propositional variable $p \in Z$. Then, $k(\alpha) \leq f(\alpha) \leq K(\alpha)$ for every formula $\alpha \in Z$.

Thus, k and K can be regarded as a *lower* and an *upper approximation*, respectively, of a valuation f .

2.3. Maehara's Lemma

Lemma (Maehara's Lemma)

For all formulas $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$, if

$$g(\alpha_1) \cap \dots \cap g(\alpha_m) \leq g(\beta_1) \cup \dots \cup g(\beta_n)$$

holds for every valuation g (over Ω) on a modal algebra $\mathbf{A}$, then

$$(*) \quad k(\alpha_1) \cap \dots \cap k(\alpha_m) \leq K(\beta_1) \cup \dots \cup K(\beta_n)$$

holds for every quasi-valuation (k, K) over any Z on $\mathbf{A}$ such that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Z$.

Proof. For a given (k, K) on $\mathbf{A}$, take any valuation g on $\mathbf{A}$ satisfying $k(p) \leq g(p) \leq K(p)$ for any variable $p \in Z$. By quasi-valuation lemma, $k(\gamma) \leq g(\gamma) \leq K(\gamma)$ for every formula $\gamma \in Z$. From our assumption,

$$g(\alpha_1) \cap \dots \cap g(\alpha_m) \leq g(\beta_1) \cup \dots \cup g(\beta_n).$$

Therefore,

$$\begin{aligned} k(\alpha_1) \cap \dots \cap k(\alpha_m) &\leq g(\alpha_1) \cap \dots \cap g(\alpha_m) \\ &\leq g(\beta_1) \cup \dots \cup g(\beta_n) \leq K(\beta_1) \cup \dots \cup K(\beta_n). \end{aligned}$$

Corollary

Suppose that **S** is a sequent system for a modal logic **M**. Then, (1) implies (2).

- (1) $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in **S**,
- (2) $k(\alpha_1) \cap \dots \cap k(\alpha_m) \leq K(\alpha_1) \cap \dots \cap K(\alpha_m)$
holds for every quasi-valuation (k, K) on any **M**-algebra **A**
over any Z such that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Z$.

Corollary

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- (2) $k(\alpha_1) \cap \dots \cap k(\alpha_m) \leq K(\alpha_1) \cap \dots \cap K(\alpha_m)$
holds for every quasi-valuation (k, K) on any $\mathbf{M}$ -algebra $\mathbf{A}$
over any Z such that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Z$.

If (2) implies the following (3), then cut elimination holds for $\mathbf{S}$.

- (3) $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{S}^-$.

Thus, we have the following definition.

3. Semi-completeness

Definition (Semi-completeness)

A sequent system $\mathbf{T}$ is **semi-complete** w.r.t. a class $\mathcal{C}$ of $\mathbf{M}$ -algebras, when for all formulas $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$,

- if the inequality

$$(*) \quad k(\alpha_1) \cap \dots \cap k(\alpha_m) \leq K(\beta_1) \cup \dots \cup K(\beta_n)$$

holds for each $\mathbf{M}$ -algebra $\mathbf{A} \in \mathcal{C}$ and each quasi-valuation (k, K) on $\mathbf{A}$ over any Z such that $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in Z$,

- then the sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in $\mathbf{T}$.

3.1. Semi-completeness and cut elimination

Obviously, if $\mathbf{S}^-$ is semi-complete then cut elimination holds for $\mathbf{S}$.

♣ *In practice, taking the contraposition, we will look for (or, construct) such (k, K) and $\mathbf{A}$ that $(*)$ fails for each sequent which is not provable in $\mathbf{S}^-$.*

This is the essence of Maehara's idea.

- In fact, if cut elimination holds for $\mathbf{S}$ then $\mathbf{S}^-$ is semi-complete w.r.t. the class $\mathcal{C}$ of all $\mathbf{M}$ -algebras, when $\mathbf{S}$ is a system for a modal logic $\mathbf{M}$.
- In general, if a sequent system $\mathbf{T}$ is complete w.r.t. a class $\mathcal{C}$, then it is semi-complete w.r.t. $\mathcal{C}$.

4. A comparison with existing algebraic proofs

Here are some references to papers on algebraic proofs of cut elimination, in addition to [Mae].

- M. Okada and K. Terui (1999) — for linear logic,
- F. Belardinelli, P. Jipsen and HO (2004) for substructural and modal logics: *Algebraic aspects of cut elimination*, [BJO] *Studia Logica* 77,
- A. Ciabattoni, N. Galatos and K. Terui (2012) — algebraic proofs and MacNeille completions.

Since [BJO] was written under the influence of [Mae], it is not difficult to see its connection of the present approach.

- Except [Mae], the notion of quasi-valuations had not been used explicitly in these works. The **quasi-embedding lemma** in [BJO] is a special case of our quasi-valuation lemma. In fact, the quasi-embedding used in it corresponds to our mapping K .
- A way of constructing a required algebra $\mathbf{A}$ used in them was essentially the same. Main novelty of our [BJO] lies in a purely algebraic description of this construction, called **quasi-completion**, which turned out to be a generalization of **Dedekind-MacNeille completions**.

These algebraic arguments were applied also to sequent systems for modal and substructural predicate logics, in which **algebraic structures** of the form $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a *complete* algebra and D is a nonempty set (for individual domain) were used. Quasi-valuations on such an algebraic structure must satisfy the following;

- (7) $k(\forall x \alpha) \subseteq \bigcap \{k(\alpha[d/x]) : d \in D\}$ and $\bigcap \{K(\alpha[d/x]) : d \in D\} \subseteq K(\forall x \alpha)$
for $\forall x \alpha \in Z$,
- (8) $k(\exists x \alpha) \subseteq \bigcup \{k(\alpha[d/x]) : d \in D\}$ and $\bigcup \{K(\alpha[d/x]) : d \in D\} \subseteq K(\exists x \alpha)$
for $\exists x \alpha \in Z$.

In this way, our semi-completeness arguments work well also for sequent systems for modal and substructural predicate logics.

Notice also semi-completeness of sequent systems of simple type theory (in [Mae]) and of second-order logic (by T. Arai, 2017).

5. Semantical proofs using Kripke frames

The key issue of the present talk is to show how model-theoretic proofs can be incorporated into our present framework. Here are some references to them.

- M. Fitting (1973) — for modal and intuitionistic logics,
- O. Lahav and A. Avron (2014) — introducing a “unified semantic framework”
- HO (2015) — an early attempt to the present topic: *Semantical approach to cut elimination and subformula property in modal logic*, in: Structural Analysis of Non-Classical Logics,

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We will first explain an example of a semantical proof of cut elimination using Kripke frames for a sequent system **GS4**, which we owe to M. Takano by his unpublished note in 2000.

5.1. Canonical construction

The proof goes similarly to a standard proof of *Kripke completeness* using canonical models. Recall that **GS4**⁻ denotes the system **GS4** without the cut rule.

- A pair (Σ, Θ) of subsets Σ and Θ of Ω is (**GS4**⁻)consistent (in Ω) if any sequent of the form $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is not provable in **GS4**⁻ for $\alpha_1, \dots, \alpha_m \in \Sigma$ and $\beta_1, \dots, \beta_n \in \Theta$.
- A pair (Σ, Θ) of subsets Σ and Θ of Ω is (**GS4**⁻)saturated in Ω , if it is maximally consistent in Ω , i.e. it is consistent and moreover for any $\gamma \in \Omega \setminus (\Sigma \cup \Theta)$, neither $(\Sigma \cup \{\gamma\}, \Theta)$ nor $(\Sigma, \Theta \cup \{\gamma\})$ is consistent.

Due to lack of cut rule in **GS4**⁻, we cannot expect the following.

- If (Σ, Θ) is consistent, then for any formula γ in Ω either $(\Sigma \cup \{\gamma\}, \Theta)$ or $(\Sigma, \Theta \cup \{\gamma\})$ is consistent.

Thus, the union of $\Sigma \cup \Theta$ may not be equal to Ω for a saturated pair (Σ, Θ) .

But still we can show the following by using Zorn's lemma, as the set of all consistent pairs is inductive.

Lemma (saturation)

For every consistent pair (Σ, Θ) there exists a saturated pair (Σ^, Θ^*) such that $\Sigma \subseteq \Sigma^*$ and $\Theta \subseteq \Theta^*$.*

Define a Kripke model $\langle W, R, V \rangle$ as follows.

- W is the set of all saturated pairs (Σ, Θ) in Ω ,
- For every $(\Sigma, \Theta), (\Lambda, \Pi) \in W$, the relation $(\Sigma, \Theta)R(\Lambda, \Pi)$ holds iff $\Sigma_{\Box} \subseteq \Lambda_{\Box}$, where $\Gamma_{\Box} = \{\beta; \Box\beta \in \Gamma\}$,
- The valuation V is defined by $V(p) = \{(\Sigma, \Theta) \in W; p \in \Sigma\}$, for every propositional variable p .

We have that

- ① the structure $\langle W, R \rangle$ is a Kripke frame for **S4**,
- ② for each formula $\alpha \in \Omega$ and each $(\Sigma, \Theta) \in W$,
 - if $\alpha \in \Sigma$ then $(\Sigma, \Theta) \models \alpha$,
 - if $\alpha \in \Theta$ then $(\Sigma, \Theta) \not\models \alpha$.

(cf. semivaluations in Schütte (1960))

The above (2) can be shown inductively by the following **downward saturation** of each saturated pair (Σ, Θ) .

5.2. Downward saturation

I. The case where α is of the form $\beta \wedge \gamma$.

- if $\beta \wedge \gamma \in \Sigma$ then both β and γ are in Σ ,
- if $\beta \wedge \gamma \in \Theta$ then either β or γ is in Θ .

II. The case where α is of the form $\beta \vee \gamma$.

- if $\beta \vee \gamma \in \Sigma$ then either β or γ are in Σ ,
- if $\beta \vee \gamma \in \Theta$ then both β and γ are in Θ .

III. The case where α is of the form $\neg\beta$.

- if $\neg\beta \in \Sigma$ then β is in Θ ,
- if $\neg\beta \in \Theta$ then β is in Σ .

IV. The case where α is of the form $\Box\beta$.

- if $\Box\beta \in \Sigma$ then $\beta \in \Lambda$ for each (Λ, Π) such that $(\Sigma, \Theta)R(\Lambda, \Pi)$,
- if $\Box\beta \in \Theta$ then $\beta \in \Pi$ for some (Λ, Π) such that $(\Sigma, \Theta)R(\Lambda, \Pi)$.

Theorem (cut elimination – with a model-theoretic proof)

*If a sequent $\Gamma \Rightarrow \Delta$ is provable in **GS4**, there exists a proof of $\Gamma \Rightarrow \Delta$ in **GS4** without any application of cut rule.*

Proof. Consider the contraposition. If $\Gamma \Rightarrow \Delta$ is not provable in **GS4**[−] then there exists a saturated pair (Σ, Θ) such that $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Theta$. Then in our Kripke model $\langle W, R, V \rangle$, we have that $(\Sigma, \Theta) \models \alpha$ for all $\alpha \in \Gamma$ and $(\Sigma, \Theta) \not\models \beta$ for all $\beta \in \Delta$. Therefore, $\Gamma \Rightarrow \Delta$ cannot be valid, and hence is not provable in **GS4**.

6. In terms of semi-completeness

We will transform the above proof into a proof of semi-completeness. Recall that W is the set of all saturated pairs. Then, the power set $\wp(W)$ with $\Box_R$ forms a modal algebra $\mathbf{A}^*$, which is in fact an **S4**-algebra. Here, $\Box_R S$ for $S(\subseteq W)$ is defined by the set

$$\{(\Sigma, \Theta) : \text{for each } (\Lambda, \Pi), \text{ if } (\Sigma, \Theta)R(\Lambda, \Pi) \text{ then } (\Lambda, \Pi) \in S\}.$$

Define (k, K) on $\mathbf{A}^*$ by

- $k(\alpha) = \{(\Sigma, \Theta) : \alpha \in \Sigma\},$
- $K(\alpha) = \{(\Sigma, \Theta) : \alpha \notin \Theta\}.$

By **downward saturation** of each $(\Sigma, \Theta) \in W$, (k, K) is shown to be a quasi-valuation.

► What does each (k, K) mean from model-theoretic and proof-theoretic viewpoint?

- Downward saturation of antecedent and succedent of each saturated pair,
- Admissibility of left and right rules for each logical connective.

[Observation due to Takano]

Lemma (Semi-completeness of **GS4**[−])

*Assume that $k(\alpha_1) \cap \dots \cap k(\alpha_m) \subseteq K(\beta_1) \cup \dots \cup K(\beta_n)$ holds in **A**^{*}. Then the sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in **GS4**[−].*

Proof. Our assumption implies that for any $(\Sigma, \Theta) \in W$, if $\alpha_i \in \Sigma$ for all i then $\beta_k \notin \Theta$ for some k . Suppose that the above sequent is not provable in **GS4**[−]. Then, there exists $(\Sigma^*, \Theta^*) \in W$ such that $\alpha_i \in \Sigma^*$ for all i and also $\beta_j \in \Theta^*$ for all j . But this contradicts our assumption.

Similar arguments work for some other modal logics and also for a multiple-succedent system **LJ**' for intuitionistic logic.

7. A further extension

We can extend “model-theoretic proofs” of cut elimination to proofs for sequent systems for **modal predicate logics**, including the predicate extension **GQS4** of **GS4**, and also for **intuitionistic predicate logic QLJ'**.

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We can extend “model-theoretic proofs” of cut elimination to proofs for sequent systems for **modal predicate logics**, including the predicate extension **GQS4** of **GS4**, and also for **intuitionistic predicate logic QIJ'**.

- This can be carried out by introducing the notion of **Henkin saturations** with varying domains.
- Then, we can show **Henkin saturation lemma**, a stronger form of saturation lemma. Using the set of all “Henkin saturated triples”, instead of saturated pairs, we get a required Kripke model with varying domains.

So far so good. But if we want to transform this proof into semi-completeness, we will face some technical difficulties since it is necessary to construct algebraic structures corresponding to Kripke frames with varying domains. Then how?

Notice that the present problem is closely related to the fact that MacNeille completions preserve existing infinite joins and meets, while canonical extensions don't always.

7.1. Expanded algebraic structures

To overcome this problem, we introduce **expanded algebraic structures**. The triple $\langle \mathbf{A}, D, \phi \rangle$ is an expanded algebraic structure for modal (intuitionistic) predicate logic if

- $\mathbf{A}$ is a complete modal (Heyting, resp.) algebra,
- D is a nonempty set,
- ϕ is a mapping from D to A satisfying that $\bigcup \{ \phi(d) : d \in D \} = 1_{\mathbf{A}}$, (and moreover $\phi(d) \leq \Box \phi(d)$ for each $d \in D$ for a modal structure.)

Valuations over expanded algebraic structures are defined similarly to those over usual algebraic structures, except

- $f(\forall x \alpha) = \bigcap \{ \phi(d) \rightarrow f(\alpha[d/x]) : d \in D \},$
- $f(\exists x \alpha) = \bigcup \{ \phi(d) \wedge f(\alpha[d/x]) : d \in D \}.$

The validity of a formula β containing free variables is defined by the validity of the universal closure of β .

Lemma (Completeness of $\mathbf{QLJ}'$ w.r.t. expanded structures)

A sequent is provable in $\mathbf{QLJ}'$ iff it is valid in every expanded algebraic structure for intuitionistic predicate logic.

Our expanded algebraic structures for modal (intuitionistic) predicate logic(s) are simplified versions of **modal valued structures** (and **Heyting-valued structures**, resp.) in;

- D. Gabbay, V. Shehtman and D. Skvortsov, *Quantification in Nonclassical Logic I*, Elsevier (2009)

7.2. Semi-completeness w.r.t. expanded structures

Quasi-valuations can be naturally extended to those over expanded algebraic structures. Now, model-theoretic proofs mentioned above can be transformed into our framework. Thus,

Theorem (Semi-completeness w.r.t. expanded structures)

The sequent system $\mathbf{QLJ}'^-$ is semi-complete w.r.t. expanded algebraic structures for intuitionistic predicate logic. Similarly for $\mathbf{GQS4}^-$.

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Quasi-valuations can be naturally extended to those over expanded algebraic structures. Now, model-theoretic proofs mentioned above can be transformed into our framework. Thus,

Theorem (Semi-completeness w.r.t. expanded structures)

The sequent system $\mathbf{QLJ}'^-$ is semi-complete w.r.t. expanded algebraic structures for intuitionistic predicate logic. Similarly for $\mathbf{GQS4}^-$.

Some time later I found that the above theorem for $\mathbf{QLJ}'^-$ was essentially proved already in Part 3: “Algebraic Models” of;

- A.G. Dragalin, *Mathematical Intuitionism: Introduction to Proof Theory*, AMS (1988).

★ The framework due to Maehara can cover many of existing standard semantical proofs of cut elimination whether algebraic ones or model-theoretic, by using an algebraic construction which is a generalization of either MacNeille completions or complex algebras.